MOREILLON Philippe NT2940 Last Session

Probability spaces and random variables

* Algebra : Let Ω be a set. $\mathcal{A} \subset \mathcal{P}(\Omega)$ is an algebra on Ω if 1./ $\Omega \in \mathcal{A}$
2./ $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3./ $A \in \mathcal{A}, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

* σ-algebra : Let Ω be a set. $\mathcal{F} \subset \mathcal{P}(\Omega)$ is a σ-algebra on Ω if 1./ $\Omega \in \mathcal{F}$
2./ $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3./ $A_n \in \mathcal{F}\ \forall n \geq 1 \Rightarrow \bigcup_{n \geq 1} A_n \in \mathcal{F}$.

* Borel σ-algebra : $\mathcal{B}(\mathbb{R}) = \sigma\{(-\infty, a) ; a \in \mathbb{R}\}$ - the σ-algebra generated by the sets $(-\infty, a)$; $a \in \mathbb{R}$

* Measure : Let Ω be a set and $\mathcal{F} \subset \mathcal{P}(\Omega)$ a σ-algebra on Ω. A function $\mu : \mathcal{F} \to \mathbb{R}_+$ is called a measure if
(1.) $\mu(\emptyset) = 0$
(2.) $\forall$ collection $\{A_i\}_{i \geq 1}$; $A_i \in \mathcal{F}\ \forall i \geq 1$ and $A_i \cap A_j = \emptyset\ \forall i \neq j$, $\mu\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} \mu(A_i)$ (countable-additivity)

If μ also verifies $\mu(\Omega) = 1$, μ is called a probability measure, and $(\Omega, \mathcal{F}, \mu)$ a probability space.

Properties of a probability measure : (1) subadditivity : If $A \subset \bigcup_{i \geq 1} A_i$, $\mathbb{P}(A) \leq \sum_{i \geq 1} \mathbb{P}(A_i)$
Let $\mathbb{P}$ be a probability measure. (2) continuity from below : If $A_1 \subset A_2 \subset \ldots$, $\mathbb{P}\left(\bigcup_{i \geq 1} A_i\right) = \lim_i \mathbb{P}(A_i)$
(3) continuity from above : If $A_1 \supset A_2 \supset \ldots$, $\mathbb{P}\left(\bigcap_{i \geq 1} A_i\right) = \lim_i \mathbb{P}(A_i)$

* Random variable : Let Ω be a set and $\mathcal{F}$ a σ-algebra on Ω. $X : \Omega \to \mathbb{R}$ is a random variable if $\forall A \in \mathcal{B}(\mathbb{R})$, $X^{-1}(A) := \{\omega \in \Omega ; X(\omega) \in A\} \in \mathcal{F}$. This condition is called measurability.

We call $\sigma(X) := \sigma\{X^{-1}(A) ; A \in \mathcal{B}(\mathbb{R})\}$ the σ-algebra generated by X.

* Distribution functions : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $\begin{array}{l} \mathbb{R} \to [0,1] \\ x \mapsto F_X(x) := \mathbb{P}(X \leq x) \end{array}$ is called distribution function of X.

Properties : (1) F_X is non-decreasing.
(2) $\lim_{x \to +\infty} F_X(x) = 1$ and $\lim_{x \to -\infty} F_X(x) = 0$.
(3) F_X is right-continuous
(4) $\lim_{y \uparrow x} F_X(y) = \mathbb{P}(X < x)$
(5) $\mathbb{P}(X = x) = F_X(x) - \lim_{y \uparrow x} F_X(y)$

* independence : $A, B \in \mathcal{F}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Two random variables X, Y are independent if $\forall A, B \in \mathcal{B}(\mathbb{R})$, $X^{-1}(A)$ and $Y^{-1}(B)$ are independent.

* Expectation : if X is discrete, $\mathbb{E}[X] = \sum_k \mathbb{P}(X = k) \cdot k$
if X is continuous and $f_X(x) = \frac{d}{dx} F_X(x)$, $\mathbb{E}[X] = \int x f_X(x) dx$

Remark : if $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable, $\mathbb{E}[\varphi(X)] = \begin{cases} \sum_k \varphi(k) \mathbb{P}(X = k) & \text{if } X \text{ discrete} \\ \int \varphi(x) f_X(x) dx & \text{if } X \text{ continuous} \end{cases}$

* Variance : $Var(X) = \mathbb{E}[(X-\mathbb{E}[X])^2]$

Determining distributions and dependence :

Let $X, Y, Z \in \mathcal{N}(0,1)$ be independent. What are the techniques to compute f_{X-Y}, f_{Y-Z} and f_{Z-X} ? What are the techniques to show $X-Y$, $Y-Z$ and $Z-X$ are independent of $X+Y+Z$? (without using tools from the study of Gaussian vectors)

1/ Direct Method :

$$f_{X-Y}(u) = \frac{d}{du}\mathbb{P}(X-Y \le u) = \frac{d}{du}\mathbb{P}((x,y) : x-y \le u) \underset{X,Y \text{ ind}}{=} \frac{d}{du}\int_{-\infty}^{+\infty}\int_{-\infty}^{u+y} f_X(x) f_Y(y)\,dx\,dy = \int_{-\infty}^{+\infty} f_Y(y) f_X(u+y)\,dy$$

$$\Rightarrow f_{X-Y}(u) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-(u+y)^2/2}\,dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\left(\frac{u^2}{2}+uy+y^2\right)}dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{u^2}{4}} \cdot e^{-\frac{u^2}{2}} \cdot e^{-\left(\frac{u^2}{4}+uy+y^2\right)}dy$$

$$= \frac{1}{2\pi} e^{-u^2/4} \underbrace{\int_{-\infty}^{+\infty} e^{-(y+\frac{u}{2})^2}dy} = \frac{1}{2\sqrt{\pi}} e^{-u^2/4} \quad \Rightarrow X-Y \in \mathcal{N}(0,2)$$

if $\tilde{X} \in \mathcal{N}(-\frac{u}{2}, \frac{1}{2})$, then $f_{\tilde{X}}(x) = \frac{\sqrt{2}}{\sqrt{2\pi}} e^{-(x+u/2)^2} \Rightarrow \int_{-\infty}^{+\infty} e^{-(y+\frac{u}{2})^2}dy = \sqrt{\pi}$

2/ Via characteristic functions : $\varphi_{X-Y}(t) = \mathbb{E}[e^{it(X-Y)}] = \mathbb{E}[e^{itX} \cdot e^{-itY}] \underset{X,Y \text{ ind}}{=} \mathbb{E}[e^{itX}] \cdot \mathbb{E}[e^{-itY}] = (e^{-t^2/2})^2 = e^{-t^2}$

$\Rightarrow X-Y \in \mathcal{N}(0,2)$.

Now if we want to study the dependence :

1/ Change of variables Method : Let $u_1, u_2 \in \mathbb{R}$,

$$\mathbb{P}(X-Y \le u_1, X+Y+Z \le u_2) = \mathbb{P}(X-Y \le u_1, X+Y \le u_2 - Z, Z \in \mathbb{R}) = \mathbb{P}((X,Y) \in F((-\infty, u_1], (-\infty, u_2 - Z]), Z \in \mathbb{R})$$

We want F ; $F(X-Y, X+Y) = (X,Y)$

$$\begin{cases} U = X-Y \\ V = X+Y \end{cases} \Leftrightarrow \begin{cases} X = \frac{U+V}{2} \\ Y = \frac{V-U}{2} \end{cases} \Rightarrow F(U,V) = \left(\frac{U+V}{2}, \frac{V-U}{2}\right)$$

$$\underset{X,Y,Z \text{ ind}}{=} \int_{-\infty}^{+\infty}\iint_{F((-\infty,u_1],(-\infty,u_2])} f_X(x) f_Y(y) f_Z(z)\,dx\,dy\,dz$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{u_2 - z}\int_{-\infty}^{u_1} f_X\left(\frac{u+v}{2}\right) f_Y\left(\frac{v-u}{2}\right) f_Z(z)\,\frac{1}{2}\,du\,dv\,dz$$

We want to do a change of variables to retrieve $X-Y$ and $X+Y$.

$$\begin{cases} U = X-Y \\ V = X+Y \end{cases} \quad \begin{pmatrix} dx/du & dx/dv \\ dy/du & dy/dv \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \Rightarrow |\det| = \frac{1}{2}$$

$$\text{Now since } \mathbb{P}(X-Y \le u_1, X+Y+Z \le u_2) = \mathbb{P}(X-Y \le u_1, X+Y \le u_2 - Z, Z \in \mathbb{R}) \underset{X,Y,Z \text{ ind}}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{u_2 - z}\int_{-\infty}^{u_1} f_{(X-Y, X+Y)}(u,v) \cdot f_Z(z)\,du\,dv\,dz$$

We can conclude that $f_{(X-Y, X+Y)}(u,v) = f_X\left(\frac{u+v}{2}\right) f_Y\left(\frac{v-u}{2}\right) \cdot \frac{1}{2}$ (since $\int_{-\infty}^{u}\int_{-\infty}^{v} g_1(x,y)\,dx\,dy = \int_{-\infty}^{u}\int_{-\infty}^{v} g_2(x,y)\,dx\,dy \ \forall u,v \Rightarrow g_1 = g_2$)

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{v-u}{2}\right)^2} \cdot \frac{1}{2} = \frac{1}{4\pi} e^{-\frac{1}{4}(u^2+v^2)}$$

$$\Rightarrow \mathbb{P}(X-Y \le u_1, X+Y+Z \le u_2) = \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{u_2 - z} \frac{1}{4\pi} e^{-\frac{1}{4}v^2}\,dv\,dz}_{=:\,g_1(u_2)} \cdot \underbrace{\int_{-\infty}^{u_1} e^{-1/4\,u^2}\,du}_{=:\,g_2(u_1)}$$

Since $\mathbb{P}(X-Y \le u_1, X+Y+Z \le u_2)$ factorizes in the product of two functions of u_1 and u_2 respectively, $X-Y$ and $X+Y+Z$ are independent.

Remark : via this technique we obtained the independence by obtaining the joint probability distribution function.

2./ characteristic functions method :

$$\varphi_{(X-Y,\,X+Y+Z)}(s,t) = \mathbb{E}\left[e^{is(X-Y)+it(X+Y+Z)}\right] = \mathbb{E}\left[e^{i(s+t)X+i(t-s)Y+itZ}\right] \underset{X,Y,Z \text{ ind.}}{=} \mathbb{E}\left[e^{i(s+t)X}\right]\mathbb{E}\left[e^{i(t-s)Y}\right]\mathbb{E}\left[e^{itZ}\right]$$

$$= e^{-(s+t)^2/2} \cdot e^{-(t-s)^2/2} \cdot e^{-t^2/2} = e^{-s^2} \cdot e^{-t^2} \cdot e^{-t^2/2} = \underbrace{e^{-s^2}}_{g_1(s)} \cdot \underbrace{e^{-\frac{3}{2}t^2}}_{g_2(t)}$$

$\Rightarrow$ Since $\varphi_{(X-Y,\,X+Y+Z)}(s,t)$ factorizes into the product of two functions depending on s and t respectively,

$X-Y$ and $X+Y+Z$ are independent.

<u>Conditional probability and expectation</u>

* conditional probability : Let $B \in \mathcal{F}$; $\mathbb{P}(B) > 0$. $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ $\forall A \in \mathcal{F}$.

* conditional expectation : Let $B \in \mathcal{F}$; $\mathbb{P}(B) > 0$. $\mathbb{E}[X|B] = \dfrac{\mathbb{E}[X \mathbb{1}_B]}{\mathbb{P}(B)}$ $\forall$ random variable X.

$\rightsquigarrow$ conditioning with respect to a partition. Let $\{A_i\}_{i\geq 1}$ be such that $A_i \in \mathcal{F}$ $\forall i \geq 1$ and $A_i \cap A_j = \emptyset$ $\forall i \neq j$, and $\mathbb{P}(A_i) > 0$ $\forall i \geq 1$. Let $\sigma(\{A_i\}_{i\geq 1})$ denote the σ-algebra generated by $\{A_i\}_{i\geq 1}$, and let X be a random variable.
$\mathbb{E}[X|\sigma(\{A_i\}_{i\geq 1})] := \sum_{i\geq 1} \mathbb{E}[X|A_i] \cdot \mathbb{1}_{A_i}$: a random variable called conditional expectation of X w.r.t. $\{A_i\}_{i\geq 1}$.

Properties : (1.) $\mathbb{E}[X|\sigma(\{A_i\}_{i\geq 1})]$ is $\sigma(\{A_i\}_{i\geq 1})$ - measurable.

(2.) $\forall B \in \sigma(\{A_i\}_{i\geq 1})$, $\mathbb{E}[X \mathbb{1}_B] = \mathbb{E}[\mathbb{E}[X|\sigma(\{A_i\}_{i\geq 1})]\, \mathbb{1}_B]$

(3.) If Y is $\sigma(\{A_i\}_{i\geq 1})$ - measurable, $\mathbb{E}[XY|\sigma(\{A_i\}_{i\geq 1})] = Y \cdot \mathbb{E}[X|\sigma(\{A_i\}_{i\geq 1})]$

In particular, $\mathbb{E}[Y|\sigma(\{A_i\}_{i\geq 1})] = Y$ whenever Y is $\sigma(\{A_i\}_{i\geq 1})$ - measurable.

Via approximations, we can define $\mathbb{E}[X|\mathcal{S}]$ for any σ-algebra $\mathcal{S} \subset \mathcal{F}$, but $\mathbb{E}[X|\mathcal{S}]$ cannot be expressed explicitely as before. It is determined by the following property : $\mathbb{E}[\mathbb{E}[X|\mathcal{S}]\mathbb{1}_B] = \mathbb{E}[X \mathbb{1}_B]$ $\forall B \in \mathcal{S}$.

It also verifies : (1.) $\mathbb{E}[\alpha X + \beta Y|\mathcal{S}] = \alpha \mathbb{E}[X|\mathcal{S}] + \beta \mathbb{E}[Y|\mathcal{S}]$ $\forall \alpha, \beta \in \mathbb{R}$

(2.) $\mathbb{E}[\mathbb{E}[X|\mathcal{S}]] = \mathbb{E}[X]$

(3.) $\mathbb{E}[YX|\mathcal{S}] = Y\, \mathbb{E}[X|\mathcal{S}]$ $\forall Y$ $\mathcal{S}$-measurable

(4.) $\mathbb{E}[X|\mathcal{S}] = \mathbb{E}[X]$ if X is independent of $\mathcal{S}$

(5.) $\mathbb{E}[\mathbb{E}[X|\mathcal{S}_2]|\mathcal{S}_1] = \mathbb{E}[X|\mathcal{S}_1]$ if $\mathcal{S}_1 \subset \mathcal{S}_2$ are σ-algebras.

(6.) If $X \geq 0$, $\mathbb{E}[X|\mathcal{S}] \geq 0$.

Exercise : Let X, Y be two $Exp(\alpha)$ independent random variables, with α such that $\mathbb{E}[\log(X+Y)] = 1$.

(1) To show $\frac{X}{Y}$ and $X+Y$ are independent

(2) compute $\mathbb{E}[\log(X) | \log(\frac{X}{Y})]$

(1) Homework.

(2) Idea : Use (1) to rewrite $\log(X)$ in terms of $\frac{X}{Y}$ and $X+Y$ and use the properties of conditional expectation :

$$\begin{cases} U = \frac{X}{Y} \\ V = X+Y \end{cases} \Leftrightarrow \begin{cases} U = \frac{X}{Y} \\ X = V - Y \end{cases} \Leftrightarrow \begin{cases} U = \frac{V-Y}{Y} \\ X = V-Y \end{cases} \Leftrightarrow \begin{cases} Y = \frac{V}{1+U} \\ X = V - \frac{V}{1+U} = \frac{UV}{1+U} \end{cases}$$

$$\Rightarrow X = \frac{UV}{1+U} = \frac{X}{Y} \cdot (X+Y) \cdot \frac{1}{1+\frac{X}{Y}}$$

Hence $\mathbb{E}[\log(X) | \log(\frac{X}{Y})] = \mathbb{E}[\log(\frac{X}{Y} \cdot (X+Y) \cdot \frac{1}{1+\frac{X}{Y}}) | \log(\frac{X}{Y})] = \mathbb{E}[\log(\frac{X}{Y}) + \log(X+Y) - \log(1+\frac{X}{Y}) | \log(\frac{X}{Y})]$

$\overset{lin.}{=} \mathbb{E}[\log(\frac{X}{Y}) | \log(\frac{X}{Y})] + \mathbb{E}[\log(X+Y) | \log(\frac{X}{Y})] - \mathbb{E}[\log(1+\frac{X}{Y}) | \log(\frac{X}{Y})]$

$= \log(\frac{X}{Y}) + \mathbb{E}[\log(X+Y)] - \log(1+\frac{X}{Y}) = \log(\frac{X}{X+Y}) + \mathbb{E}[\log(X+Y)] = \underline{1 + \log(\frac{X}{X+Y})}.$

measurability + independence

* Conditional probability and expectation in the continuous case.

if (X,Y) has a p.d.f $f(x,y)$, then define $f_{X|Y=y}(x) := \frac{f_{(X,Y)}(x,y)}{f_Y(y)}$ and $\mathbb{E}[X|Y=y] := \int_{-\infty}^{+\infty} x f_{X|Y=y}(x)\,dx$

Then $y \mapsto \mathbb{E}[X|Y=y]$ satisfies all six properties of the conditional expectation $\mathbb{E}[X|\sigma(Y)]$.

Hence $\mathbb{E}[X|Y=y] = \mathbb{E}[X|\sigma(Y)](y)$ a.s.

<u>Transforms of random variables</u>

* Characteristic function : Let X be a random variable. $t \mapsto \varphi_X(t) := \mathbb{E}[e^{itX}]$ $(\mathbb{R} \to \mathbb{C})$ is the characteristic function of X.

More generally, $(s,t) \mapsto \varphi_{(X,Y)}(s,t) = \mathbb{E}[e^{i(sX+tY)}]$.

Properties : (1) $X \overset{d}{=} Y \Leftrightarrow \varphi_X(t) = \varphi_Y(t)\ \forall t \in \mathbb{R}$.

(2) X and Y are independent $\Leftrightarrow \varphi_{(X,Y)}(s,t) = \varphi_X(s)\varphi_Y(t)\ \forall s,t \in \mathbb{R}$

(3) if $\mathbb{E}[X^n] < \infty$, $\varphi_X(t) = \sum_{k=0}^{n} \mathbb{E}[X^k] \frac{(it)^k}{k!} + o(|t|^n)$, $n \geq 1$.

In particular, $i\mathbb{E}[X] = \frac{d}{dt}\varphi_X(0)$

* Generating function : Let X be a discrete random variable. $t \mapsto g_X(t) := \mathbb{E}[t^X]$ is the probability generating function of X.

Properties : (1) $g_X(t) = g_Y(t)\ \forall t \Leftrightarrow X \overset{d}{=} Y$.

(2) $g_{X_1+X_2+\dots+X_n}(t) = g_{X_1}(t) g_{X_2}(t) \cdots g_{X_n}(t)$ if $X_1, \dots, X_n$ are independent.

* Moment generating functions : let X be a random variable. $t \mapsto \psi_X(t) := \mathbb{E}[e^{tX}]$ is the moment generating function of X.

Properties : (1) $\psi_X(t) = \psi_Y(t)\ \forall t \in Dom(\psi_X) \cap Dom(\psi_Y) \Rightarrow X \overset{d}{=} Y$

(2) $\psi_{S_n}(t) = \psi_{X_1}(t) \cdot \ldots \cdot \psi_{X_n}(t)\quad \forall t \in \bigcap_{i=1}^{n} Dom(\psi_{X_i})$, if $X_1, \ldots, X_n$ are independent

(3) if $Dom(\psi_X) \neq \{0\}$, $\mathbb{E}[X^K] < \infty\ \forall K > 0$ and $\mathbb{E}[X^K] = \psi_X^{(K)}(0)$.

Convergence of sequences of random variables.

Let $(X_n)_{n\geq 1}$ be a sequence of random variables and X a random variable.

* Almost sure convergence : $X_n \xrightarrow{a.s} X$ as $n \to +\infty$ if $\mathbb{P}(\{\omega \in \Omega ; \lim_n X_n(\omega) = X(\omega)\}) = 1$.

Properties : $X_n \xrightarrow{a.s} X$ and $Y_n \xrightarrow{a.s} Y \Rightarrow \begin{cases} X_n + Y_n \xrightarrow{a.s} X + Y \\ X_n - Y_n \xrightarrow{a.s} X - Y \\ X_n \cdot Y_n \xrightarrow{a.s} X \cdot Y \\ \frac{X_n}{Y_n} \xrightarrow{a.s} \frac{X}{Y} \quad \text{if } Y \neq 0 \text{ a.s.} \end{cases}$

* Convergence in probability : $X_n \xrightarrow{\mathbb{P}} X$ as $n \to +\infty$ if $\forall \varepsilon > 0\ \mathbb{P}(|X_n - X| > \varepsilon) \to 0$ as $n \to +\infty$.

Properties : $X_n \xrightarrow{a.s} X \Rightarrow X_n \xrightarrow{\mathbb{P}} X$

$X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y \Rightarrow \begin{cases} X_n + Y_n \xrightarrow{\mathbb{P}} X + Y \\ X_n - Y_n \xrightarrow{\mathbb{P}} X - Y \\ X_n Y_n \xrightarrow{\mathbb{P}} XY \\ \frac{X_n}{Y_n} \xrightarrow{\mathbb{P}} \frac{X}{Y} \quad \text{if } Y \neq 0 \text{ a.s.} \end{cases}$

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, $X_n \xrightarrow{\mathbb{P}} X \Rightarrow g(X_n) \xrightarrow{\mathbb{P}} g(X)$.

* Convergence in distribution : $X_n \xrightarrow{d} X$ as $n \to +\infty$ if $F_{X_n}(x) \to F_X(x)\ \forall x$ such that F_X is continuous at x.

Properties : $X_n \xrightarrow{\mathbb{P}} X \Rightarrow X_n \xrightarrow{d} X$.

$X_n \xrightarrow{d} X \Leftrightarrow \psi_{X_n}(t) \to \psi_X(t)\ \forall t \in \mathbb{R}$.

Let $c \in \mathbb{R}$ be a constant. $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{\mathbb{P}} c$

Crámer-Slutsky : $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{\mathbb{P}} Y \Rightarrow \begin{cases} X_n + Y_n \xrightarrow{d} X + Y \\ X_n - Y_n \xrightarrow{d} X - Y \\ X_n Y_n \xrightarrow{d} XY \\ \frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{Y} \quad \text{if } Y \neq 0 \text{ a.s.} \end{cases}$

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Let X be such that $f_X(x) = \begin{cases} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), & x > 0 \\ 0 & \text{otherwise} \end{cases}$ and $c > 0$.

What is the distribution of $\frac{X}{c}$?

Technique 1 :

$$F_{\frac{X}{c}}(x) = \mathbb{P}\left(\frac{X}{c} \leq x\right) = \mathbb{P}(X \leq cx) = \int_0^{cx} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \underset{\tilde{x} = \frac{x}{c}}{=} \int_0^{x} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{c^2\tilde{x}^2}{2\sigma^2}\right) c\, d\tilde{x}$$

$$\Rightarrow f_{\frac{X}{c}}(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{c^2 x^2}{2\sigma^2}\right) \cdot c .$$

Technique 2 :

$$\frac{d}{dx} F_{\frac{X}{c}}(x) = \frac{d}{dx} \int_0^{cx} \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \;\; \underset{/}{=} \;\; \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{c^2 x^2}{2\sigma^2}\right) \cdot c$$

$$\frac{d}{dx} G(cx) = G'(cx) \cdot [cx]' = G'(cx) \cdot c$$

$$\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$$

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Let (X_1,X_2) be a Gaussian vector with mean $(0,0)$ and covariance matrix $\begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}$:

Find the distribution of $\dfrac{X_1^2-2\rho X_1X_2+X_2^2}{1-\rho^2}$ by computing its moment generating function.

1/ clever way.

Notice that $\dfrac{X_1^2-2\rho X_1X_2+X_2^2}{1-\rho^2} = \dfrac{1}{1-\rho^2}(X_1\ X_2)\begin{pmatrix}1&-\rho\\ -\rho&1\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} = (X_1\ X_2)\,\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{-1}\begin{pmatrix}X_1\\ X_2\end{pmatrix}$

Since $\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}$ is symmetric, $\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{-1}$ is symmetric too and, $\exists A$; $A^T\begin{pmatrix}\lambda_1^{-1}&0\\ 0&\lambda_2^{-1}\end{pmatrix}A = \mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}$. Set $\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}=A\begin{pmatrix}X_1\\ X_2\end{pmatrix}$.

Then $(Y_1\ Y_2)\begin{pmatrix}\lambda_1^{-1}&0\\ 0&\lambda_2^{-1}\end{pmatrix}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix} = (X_1\ X_2)A^T\begin{pmatrix}\lambda_1^{-1}&0\\ 0&\lambda_2^{-1}\end{pmatrix}A\begin{pmatrix}X_1\\ X_2\end{pmatrix} = (X_1\ X_2)A^T(A^T)^{-1}\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{-1}A^{-1}A\begin{pmatrix}X_1\\ X_2\end{pmatrix}$

$= (X_1\ X_2)\,\mathrm{cov}^{-1}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}.$

$\mathrm{cov}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix} = \mathrm{cov}\left(A\begin{pmatrix}X_1\\ X_2\end{pmatrix}\right) = A\,\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}A^T$

$\Rightarrow \mathrm{cov}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}^{-1} = (A^T)^{-1}\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{-1}A^{-1}$

$\Rightarrow (X_1\ X_2)\,\mathrm{cov}^{-1}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} = \lambda_1^{-1}Y_1^2+\lambda_2^{-1}Y_2^2$

Remember : if $Z_1, Z_2, \dots, Z_n$ are iid $\mathcal{N}(0,1)$, then $\sum_{i=1}^{n} Z_i^2 \in \chi^2(n)$

Notice that $\begin{pmatrix}\alpha\\ -\rho\end{pmatrix}$ and $\begin{pmatrix}-\rho\\ \alpha\end{pmatrix}$ are linearly dependent iff $\alpha=-\rho \Rightarrow \lambda_1=\lambda_2=1+\rho$.

$\Rightarrow \lambda_1^{-1}=\lambda_2^{-1}=(1+\rho)^{-1}$

We just need to determine the law of Y_1 and Y_2 to conclude.

$\mathbb{E}\left[\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}\right] = \mathbb{E}\left[A\begin{pmatrix}X_1\\ X_2\end{pmatrix}\right] = A\,\mathbb{E}\left[\begin{pmatrix}X_1\\ X_2\end{pmatrix}\right] = \begin{pmatrix}0\\ 0\end{pmatrix}$

$\mathrm{cov}\left(\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}\right) = \mathrm{cov}\left(A\begin{pmatrix}X_1\\ X_2\end{pmatrix}\right) = A\,\mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}A^T = \begin{pmatrix}\lambda_1&0\\ 0&\lambda_2\end{pmatrix}$ by construction of A.

$\Rightarrow Y_1 \in \mathcal{N}(0,1+\rho),\ Y_2 \in \mathcal{N}(0,1+\rho)$

Now $\dfrac{1}{\sqrt{1+\rho}}Y_1 \in \mathcal{N}(0,1)$ and $\dfrac{1}{\sqrt{1+\rho}}Y_2 \in \mathcal{N}(0,1) \Rightarrow \dfrac{1}{1+\rho}Y_1^2+\dfrac{1}{1+\rho}Y_2^2 \in \chi^2(2)$

and $\lambda_1^{-1}Y_1^2+\lambda_2^{-1}Y_2^2 = \dfrac{1}{1+\rho}Y_1^2+\dfrac{1}{1+\rho}Y_2^2. \Rightarrow \dfrac{X_1^2-2\rho X_1X_2+X_2^2}{1-\rho^2} \in \chi^2(2).$

2/ Brutal way

$\mathbb{E}\left[e^{t\left(\frac{X_1^2-2\rho X_1X_2+X_2^2}{1-\rho^2}\right)}\right] = \iint_{-\infty,-\infty}^{+\infty,+\infty} e^{\frac{t(x_1^2-2\rho x_1x_2+x_2^2)}{1-\rho^2}} f_{(X_1,X_2)}(x_1,x_2)\,dx_1\,dx_2$

$= \iint_{-\infty,-\infty}^{+\infty,+\infty} e^{t\frac{(x_1^2-2\rho x_1x_2+x_2^2)}{1-\rho^2}}\dfrac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2}(x_1x_2)\,\mathrm{cov}\binom{X_1}{X_2}^{-1}\binom{x_1}{x_2}}\,dx_1\,dx_2$ $\qquad \mathrm{cov}\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{-1} = \dfrac{1}{1-\rho^2}\begin{pmatrix}1&-\rho\\ -\rho&1\end{pmatrix}$

$= \iint_{-\infty,-\infty}^{+\infty,+\infty} e^{t\frac{(x_1^2-2\rho x_1x_2+x_2^2)}{1-\rho^2}}\dfrac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}(x_1^2+x_2^2-2\rho x_1x_2)\cdot\frac{1}{1-\rho^2}}\,dx_1\,dx_2 = \iint_{-\infty,-\infty}^{+\infty,+\infty}\dfrac{1}{2\pi(1-\rho^2)^{1/2}}e^{\frac{(t-\frac{1}{2})}{1-\rho^2}(x_1^2-2\rho x_1x_2+x_2^2)}\,dx_1\,dx_2$

(*)

Now $(x_1^2 - 2\rho x_1 x_2 + x_2^2) = x_1^2 - 2\rho x_1 x_2 + \rho^2 x_2^2 - \rho^2 x_2^2 + x_2^2 = (x_1 - \rho x_2)^2 + x_2^2 (1-\rho^2)$

$$\Rightarrow (*) = \frac{1}{2\pi (1-\rho^2)^{1/2}} \iint_{-\infty}^{+\infty} e^{\frac{t-1/2}{1-\rho^2}\left((x_1-\rho x_2)^2 + x_2^2(1-\rho^2)\right)} dx_1\, dx_2$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{+\infty} e^{\frac{t-\frac{1}{2}}{1-\rho^2} x_2^2 (1-\rho^2)} \left\{ \int_{-\infty}^{+\infty} e^{\frac{t-1/2}{1-\rho^2}(x_1-\rho x_2)^2} dx_1 \right\} dx_2 \qquad (**)$$

$$e^{-\frac{1}{2}\frac{(x_1-\rho x_2)^2}{\sigma^2}} = e^{\frac{t-1/2}{1-\rho^2}(x_1-\rho x_2)^2} \Leftrightarrow -\frac{1}{2\sigma^2} = \frac{t-1/2}{1-\rho^2} \Leftrightarrow \sigma^2 = -\frac{1}{2}\frac{1-\rho^2}{t-1/2}, \quad |t| < 1/2$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{\frac{t-1/2}{1-\rho^2}(x_1-\rho x_2)^2} dx_1 = \sqrt{2\pi}\, \frac{\sqrt{1-\rho^2}}{\sqrt{1/2 - t}\,\sqrt{2}}$$

$$\Rightarrow (**) = \frac{1}{2\sqrt{\pi}\sqrt{1/2-t}} \int_{-\infty}^{+\infty} e^{(t-1/2)/1-\rho^2\, x_2^2(1-\rho^2)} dx_2 = \frac{1}{2\sqrt{\pi}\sqrt{1/2-t}} \int_{-\infty}^{+\infty} e^{(t-1/2)x_2^2} dx_2 \qquad (***)$$

$$-\frac{1}{2}\frac{x_2^2}{\sigma^2} = (t-1/2)x_2^2 \Leftrightarrow \sigma^2 = \frac{1}{2(1/2-t)} \Leftrightarrow \sigma = \frac{1}{\sqrt{2}(1/2-t)^{1/2}} \Rightarrow \int_{-\infty}^{+\infty} e^{(t-1/2)x_2^2} dx_2 = \sqrt{2\pi}\cdot\frac{1}{\sqrt{2}(1/2-t)^{1/2}}$$

$$\Rightarrow (***) = \frac{1}{2\sqrt{\pi}\sqrt{1/2-t}}\,\frac{\sqrt{2\pi}}{\sqrt{2}(1/2-t)^{1/2}} = \frac{1}{1-2t} \quad \forall t;\ |t| < 1/2$$

$$\Rightarrow \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{1-\rho^2} \sim \chi^2(2).$$

QED